International Journal of Theoretical Physics, Vol. 19, No. 5, 1980

Space-Time and Spatial Geodesics on a Rotating Sphere

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Received December 18, 1979

The motion of free particles and photons constrained to lie on the surface of a rotating sphere is analyzed. Formulas are presented for the surface area and volume of the sphere, the velocity components of free particles and photons, the time of flight between fixed reference points on the sphere, and the spatial distance along null geodesics. Spatial geodesics are also investigated and it is shown that there are many solutions of the geodesic equation joining any two fixed points. A description is given of a curved two-dimensional surface embedded in three-dimensional Euclidean space which has the same intrinsic spatial geometry as the rotating sphere.

1. INTRODUCTION

Since the earliest days of relativity theory, its application to rotating systems has been of special interest. Rotating systems provide some of the simplest examples of accelerated frames of reference, and thereby serve as a proving ground for the postulates and machinery of general relativity in the absence of permanent gravitational fields. For a review of papers published before 1962, see Arzeliès (1966); for more recent information, see for example Atwater (1970), McCrea (1971), Grøn (1975), Browne (1977), McFarlane and McGill (1978, to be referred to hereinafter as MM) and references therein.

In practice the great majority of previously published papers in this field have concentrated on the problem of the rotating disk. This may be because almost all the interest in most problems involving rotation concerns the two spatial dimensions normal to the axis of rotation, while hardly any interest attaches to the third spatial dimension. For a completely free particle, for example, the motion in the direction parallel to the axis of rotation is especially simple, and there is hardly any loss in neglecting it altogether.

There is, however, another class of problem which does not seem to have received any great attention so far, i.e., where the particles or light rays in question are constrained to lie on the surface of some rotating geometrical object. Probably the simplest and most interesting example is a rotating sphere, which is the subject of this paper, though we are currently examining other examples. Thus we shall investigate here the spatial geometry of a rotating spherical surface and the paths (in space-time) of free particles and photons moving thereon. Physically, it is easy to imagine how the necessary constraint could be imposed on a particle; while a light ray satisfying the given geometrical condition could be obtained by sending out a light pulse close to and tangential to the inner, perfectly reflecting surface of a hollow sphere.

Though some of the results are quite similar in form to those already derived in MM and elsewhere for a rotating disk, there are some major differences. First, a small rotating disk can always be regarded as part of a larger rotating disk with the same angular frequency, but the same cannot be said for rotating spherical *surfaces*. Thus although it is as well to choose a disk whose size is limited only by the requirement that no part of it should move faster than light, the results for a sphere depend on the value assigned to its radius. Second, the geometry of a rotating disk is such that the spatial and space-time geodesics exhibit no periodic properties, whereas spherical geometry gives rise, as we shall see, to repeated orbits and multiple geodesic paths connecting fixed reference points.

We shall begin by setting out some general kinematical relations between a uniformly rotating frame and a nonrotating frame, in spherical coordinates. In subsequent sections, by inserting $r = a$ throughout we shall restrict consideration to a spherical surface and to events and paths that lie thereon.

2. ELEMENTARY SPATIAL AND TEMPORAL RELATIONS

We assume that space-time is flat, and consider an inertial frame $\bar{S}$ in which reference points are identified by assigning values to spherical polar coordinates $(\bar{r}, \bar{\theta}, \bar{\phi})$, the origin of coordinates being arbitrary. If $\bar{t}$ denotes time measured in $\bar{S}$, the invariant space-time interval can be expressed as

$$ds^2 = d\bar{t}^2 - c^{-2}\left(d\bar{r}^2 + \bar{r}^2 d\bar{\theta}^2 + \bar{r}^2 \sin^2\bar{\theta}\, d\bar{\phi}^2\right) \tag{2.1}$$

Now consider the coordinate transformation

$$\bar{r} = r, \qquad \bar{\theta} = \theta, \qquad \bar{\phi} = \phi + \omega t, \qquad \bar{t} = t \tag{2.2}$$

In terms of the new coordinates (r,θ,ϕ,t),

$$ds^2 = dt^2(1-\omega^2r^2\sin^2\theta/c^2)$$
$$-c^{-2}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2+2\omega r^2\sin^2\theta\,d\phi\,dt) \qquad (2.3)$$

We note from the transformation (2.2) that the reference point (r,θ,ϕ) is described in the inertial frame $\bar{S}$ as rotating about the polar axis with constant angular velocity ω. Thus the reference system that is the set of all reference points (r,θ,ϕ) for which $r \leqslant a$, say, may be identified, provided $\omega a/c < 1$, with a sphere of radius a whose center is at $r=0$ and which rotates with angular velocity ω about the polar axis. It should be noted that our sphere may thus be minimally regarded as merely a set of rotating point particles; it is not strictly necessary to assume that it be a material continuum endowed with mechanical properties. In this way we avoid the difficulties associated with Ehrenfest's paradox.

We now apply the usual relativistic rule (Møller, 1952; Landau and Lifshitz, 1971) for the calculation of the distance dl between the neighboring reference points (r,θ,ϕ) and $(r+dr,\theta+d\theta,\phi+d\phi)$ measured in the rotating system. The result is

$$dl^2 = dr^2 + r^2\,d\theta^2 + \frac{r^2\sin^2\theta\,d\phi^2}{1-\omega^2r^2\sin^2\theta/c^2} \qquad (2.4)$$

We note that this equation implies length contraction in the ϕ direction by the factor $(1-\omega^2r^2\sin^2\theta/c^2)^{1/2}$, as would be expected by a naïve application of the special theory, but no contraction occurs in any direction normal to the local direction of motion.

It is of interest to calculate from equation (2.4) the volume and surface area of the sphere. Defining $\alpha = \omega a/c$, we have

$$V(\alpha) = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{(1-\omega^2r^2\sin^2\theta/c^2)^{1/2}}$$
$$= \frac{2\pi a^3}{\alpha^3}\left[\alpha-(1-\alpha^2)\tanh^{-1}\alpha\right] \qquad (2.5)$$

$$S(\alpha) = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{a^2\sin\theta\,d\theta\,d\phi}{(1-\alpha^2\sin^2\theta)^{1/2}} = \frac{4\pi a^2}{\alpha}\tanh^{-1}\alpha \qquad (2.6)$$

These expressions tend as expected to $\frac{4}{3}\pi a^3$ and $4\pi a^2$, respectively, as $\alpha \to 0$, and in the extreme relativistic limit $(\alpha \to 1)$ we note the curious result that $V(1) = 2\pi a^3$, but $S(\alpha) \to \infty$.

From equation (2.3), the time interval $d\tau$ between the events (r,θ,ϕ,t) and $(r,\theta,\phi,t+dt)$ according to a standard clock at rest at the reference point (r,θ,ϕ) is

$$d\tau = ds|_{d_r=0} = \gamma^{-1}dt \tag{2.7}$$

where

$$\gamma = (1-\omega^2 r^2 \sin^2\theta/c^2)^{-1/2} \tag{2.8}$$

which exhibits the expected time dilation for moving clocks. "Coordinate clocks" at rest in the rotating frame and marking the time t require to go fast by the factor γ with respect to the corresponding standard clocks at the same place.

As we have argued in MM, it is convenient, though not essential, to choose $d\tau$ as the infinitesimal time interval to be employed in the definition of velocity. If a particle moves from (r,θ,ϕ,t) to $(r+dr,\theta+d\theta,\phi+d\phi,t+dt)$, we can define its instantaneous velocity (strictly, speed) as

$$v = dl/d\tau \tag{2.9}$$

where dl and $d\tau$ are given above. In particular, its velocity components in the r, θ, and ϕ directions, respectively, are

$$v_r = \gamma\, dr/dt, \qquad v_\theta = \gamma r\, d\theta/dt, \qquad v_\phi = \gamma^2 r \sin\theta\, d\phi/dt \tag{2.10}$$

The corresponding components in the inertial frame are $\bar{v}_r = d\bar{r}/d\bar{t}$, $\bar{v}_\theta = \bar{r}\,d\bar{\theta}/d\bar{t}$ and $\bar{v}_\phi = \bar{r}\sin\bar{\theta}\,d\bar{\phi}/d\bar{t}$, and so we find from the coordinate transformation equations that

$$v_r = \gamma\bar{v}_r, \qquad v_\theta = \gamma\bar{v}_\theta, \qquad v_\phi = \gamma^2(\bar{v}_\phi - \omega r\sin\theta) \tag{2.11}$$

The speeds v and $\bar{v}$ in the two frames are related by

$$v = \gamma^2\left[\gamma^{-2}\bar{v}^2 + \omega^2 r^2 \sin^2\theta\left(1+\bar{v}_\phi^2/c^2\right) - 2\omega r\sin\theta\,\bar{v}_\phi\right]^{1/2} \tag{2.12}$$

We note that the velocity transformation equations are asymmetrical as between the two reference frames. In particular, although the velocity components of a particle in the inertial frame can never exceed c, there is no such restriction in the rotating frame.

As we shall see later, the velocity of light in the rotating frame turns out not to be equal to c, in general, so long as velocity is defined by equation (2.9); neither would it be so if we were to adopt the alternative definition $v=dl/dt$. It is possible to construct a time interval, in terms of dt and $d\phi$, for which the velocity of light is *by construction* equal to c; but, as Arzeliès (1966) shows during the corresponding analysis for a rotating disk, the synchronization condition on clocks registering this new time is extremely artificial.

3. TRAJECTORIES OF FREE PARTICLES AND PHOTONS ON A ROTATING SPHERICAL SURFACE

We restrict consideration from now on to the spherical surface $\bar{r}=r=a$. The paths followed in space-time by free particles and photons constrained to lie on this surface are given by the solutions to the appropriate geodesic equations. These equations, i.e., timelike for free particles and null for photons, may be expressed and solved in any convenient coordinate system, and the solution in any other coordinates may be obtained from the coordinate transformation equations.

Here it is simplest to work with the inertial frame coordinates $(\bar{\theta},\bar{\phi},\bar{t})$ and transform later to the rotating frame coordinates (θ,ϕ,t). Setting $\bar{r}=a$ and $d\bar{r}=0$ in equation (2.1), the timelike geodesic equation in the form

$$\frac{d}{ds}\left(g_{\mu\kappa}\frac{dx^{\mu}}{ds}\right)-\tfrac{1}{2}g_{\mu\nu,\kappa}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}=0 \tag{3.1}$$

with $\kappa=1,2,3$, gives, after solution of the trivial differential equation for $\bar{t}$,

$$\frac{d^2\bar{\theta}}{d\bar{t}^2}-\sin\bar{\theta}\cos\bar{\theta}\left(\frac{d\bar{\phi}}{d\bar{t}}\right)^2=0,\qquad \sin^2\bar{\theta}\frac{d\bar{\phi}}{d\bar{t}}=K \tag{3.2}$$

Eliminating $d\bar{\phi}/d\bar{t}$, we find after two integrations that

$$\cos\bar{\theta}=\cos\theta_0\cos\bar{v}\bar{t}/a \tag{3.3}$$

where θ_0 is the minimum value of $\bar{\theta}$ attained during the motion, $\bar{v}$ is the (constant) speed of the particle (we assume throughout that $\bar{v}\geqslant 0$) and where, without any lack of generality, we take $\bar{\theta}=\theta_0$ at $\bar{t}=0$. If $K\neq 0$, it satisfies the relation

$$K=\sigma(\bar{v}/a)\sin\theta_0 \tag{3.4}$$

where $\sigma=+1$ (-1) for an orbit which makes $\bar{\phi}$ constantly increase (decrease) with time. We then find, on integrating the equation for $d\bar{\phi}/d\bar{t}$, that

$$\sin\bar{\theta}\cos(\bar{\phi}-\phi_0)=\sin\theta_0\cos\bar{v}\bar{t}/a \tag{3.5a}$$

$$\sin\bar{\theta}\sin(\bar{\phi}-\phi_0)=\sin\sigma\bar{v}\bar{t}/a \tag{3.5b}$$

where ϕ_0 is the value of $\bar{\phi}$ at $t=0$. If, however, $K=0$, we have the singular solution

$$\cos\bar{\theta}=\cos\bar{v}\bar{t}/a, \qquad \bar{\phi}-\phi_0=0 \text{ or } \pi \tag{3.6}$$

Results for photons are obtained simply by putting $\bar{v}=c$, or equivalently by solving the null geodesic equation.

As expected, these equations describe analytically the trajectories of moving points following (at constant speed $\bar{v}$) great circle paths on the nonrotating spherical surface $\bar{r}=a$. This can be seen from application of the formulas for spherical triangles (Todhunter and Leathem, 1932) which are most conveniently expressed in the form

$$\cos\tilde{a}=\cos\tilde{b}\,\cos\tilde{c}+\sin\tilde{b}\,\sin\tilde{c}\,\cos A \tag{3.7}$$

$$\sin\tilde{a}\,\cos B=\cos\tilde{b}\,\sin\tilde{c}-\sin\tilde{b}\,\cos\tilde{c}\,\cos A \tag{3.8}$$

$$\sin\tilde{a}\,\sin B=\sin\tilde{b}\,\sin A \tag{3.9}$$

with A, B, and C denoting the three angles and $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ the angular displacements along the opposite sides. Choosing the spherical triangle whose vertices are $(\bar{\theta},\bar{\phi})$, (θ_0,ϕ_0) and the North pole $(\bar{\theta}=0)$, we see by setting $A=\pi/2$, $B=\bar{\phi}-\phi_0$, $\tilde{a}=\bar{\theta}$, $\tilde{b}=\sigma\bar{v}\bar{t}/a$ and $\tilde{c}=\theta_0$ that the equations for $\bar{\theta}$ and $\bar{\phi}$ in (3.3) and (3.5) are immediately obtained. The motion repeats itself after each complete orbit of period $2\pi a/\bar{v}$, and the locus of points traced out is

$$\tan\bar{\theta}\cos(\bar{\phi}-\phi_0)=\tan\theta_0 \tag{3.10}$$

The singular solution given by equation (3.6) corresponds to an orbit passing through the N and S poles.

We now employ the coordinate transformation in (2.2) to find the corresponding solutions in the rotating frame coordinates (θ,ϕ,t). For the general case we have

$$\cos\theta = \cos\theta_0 \cos\bar{v}t/a \tag{3.11}$$

$$\sin\theta \cos(\phi - \phi_0 + \omega t) = \sin\theta_0 \cos\bar{v}t/a \tag{3.12a}$$

$$\sin\theta \sin(\phi - \phi_0 + \omega t) = \sin\sigma\bar{v}t/a \tag{3.12b}$$

there being no change in the meaning of θ_0 and ϕ_0. For the singular case of a polar orbit, the solution becomes

$$\cos\theta = \cos\bar{v}t/a, \qquad \phi - \phi_0 + \omega t = 0 \text{ or } \pi \tag{3.13}$$

Inspection of these equations reveals that the principal effect of the rotation is to change the period in ϕ which corresponds to one complete oscillation in the θ direction. The latter still takes a time $2\pi a/\bar{v}$ (as measured by coordinate clocks), and in this time ϕ changes by $2\pi(\sigma - \omega a/\bar{v})$. It is easily seen that if $\omega a/\bar{v}$ is a rational number m/n say, then after n complete cycles in the θ direction (i.e., from θ_0 to $\pi - \theta_0$ and back, n times) the particle or photon returns to its original starting point and the motion repeats itself. The integer m is the number of rotations of the sphere with respect to the inertial frame during this time. If $\omega a/\bar{v}$ is not a rational number, the motion never repeats itself.

The locus of points traced out is found by eliminating t from equations (3.11) and (3.12). Distinguishing carefully between the two cases $\sigma = \pm 1$, we obtain

$$\phi - \phi_0 = \pm\left[\cos^{-1}(\tan\theta_0/\tan\theta) - (\sigma\omega a/\bar{v})\cos^{-1}(\cos\theta/\cos\theta_0)\right] \tag{3.14}$$

in which it is essential to interpret $\cos^{-1}$ as the nonnegative multivalued function which is zero at $\theta = \theta_0$ and which continuously increases as the particle or photon moves around the sphere. A different locus is produced by changing the sign of σ, for the same values of θ_0, ϕ_0, and $\bar{v}$, in contrast to the independence of locus on σ for a nonrotating sphere.

The velocity components v_θ and v_ϕ can now be evaluated by combining equations (2.10), (3.11), and (3.12). We find

$$v_\theta = \pm\gamma\bar{v}\left(1 - \sin^2\theta_0/\sin^2\theta\right)^{1/2}, \qquad v_\phi = \gamma^2\left(-\omega a\sin\theta + \sigma\bar{v}\sin\theta_0/\sin\theta\right) \tag{3.15}$$

where γ defined by equation (2.8) is of course evaluated at $r = a$. The

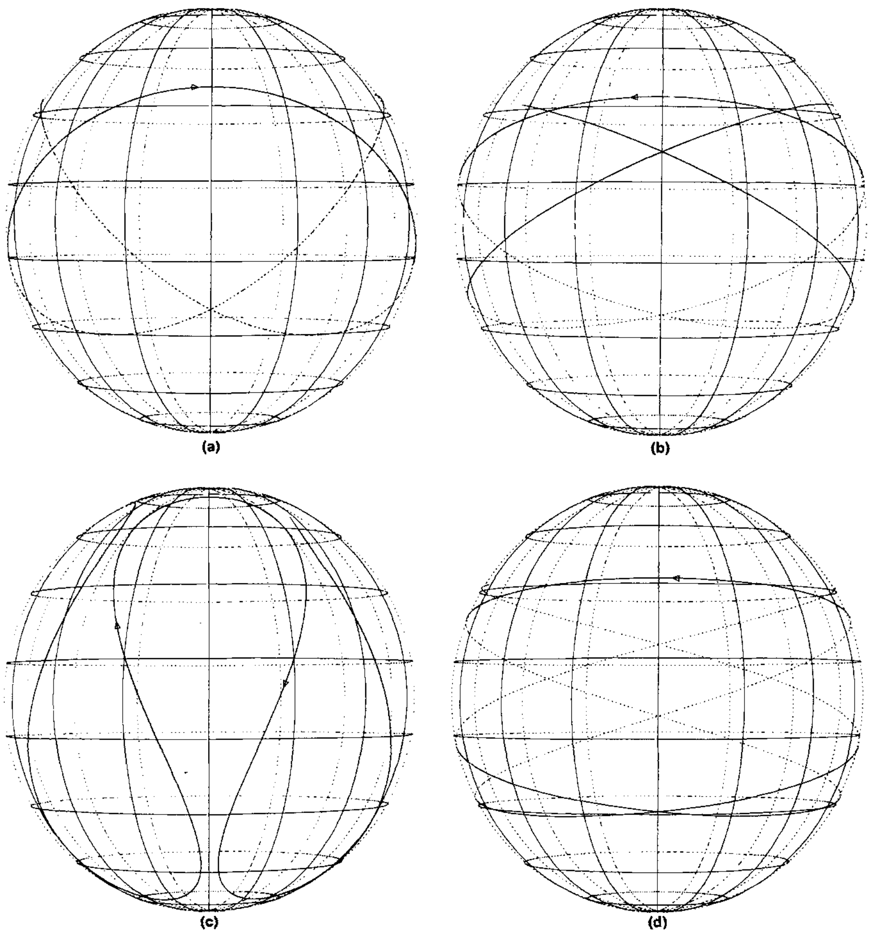

Fig. 1. Space-time geodesic arcs joining the points $A(\theta=\pi/3, \phi=-\pi/6)$ and $B(\theta=\pi/3, \phi=\pi/6)$. The four cases are (a) $A\to B(\sigma=1)$, $\omega a/\bar{v}=0.2$, (b) $B\to A(\sigma=-1)$, $\omega a/\bar{v}=0.2$, (c) $A\to B(\sigma=1)$, $\omega a/\bar{v}=0.8$, and (d) $B\to A(\sigma=-1)$, $\omega a/\bar{v}=0.8$. The lines of latitude and longitude are at 20° and $22\frac{1}{2}$° intervals, respectively.

particle's speed in the rotating frame is

$$v=\gamma^2 c\left\{\left[1-\sigma(\bar{v}/c)\alpha\sin\theta_0\right]^2-\gamma^{-2}(1-\bar{v}^2/c^2)\right\}^{1/2} \tag{3.16}$$

These results are analogous to the corresponding relations for a rotating disk [MM equations (15), (16)]. In particular, equation (3.16) with $\bar{v}=c$ confirms that the speed of light on the rotating sphere is not, in general, equal to c.

We show in Figure 1, a–d, four examples of representations of space-time geodesic paths on the surface of a rotating sphere. These are possible trajectories of particles or photons passing in each direction ($\sigma = \pm 1$) between the fixed reference points $\theta = \pi/3$, $\phi = -\pi/6$, and $\theta = \pi/3$, $\phi = \pi/6$, for two different values of the parameter $\omega a/\bar{v}$. The question of how a trajectory is determined by the coordinates of two points lying on it is postponed to Section 4, but the following observations—on how to represent paths on a rotating sphere—are relevant at this stage.

First, the geometry of the surface of a rotating sphere is inherently different from that of the surface of a nonrotating sphere, as equation (2.4) demonstrates and as we shall investigate in greater detail later. It will be appreciated, therefore, that any attempt to map the former surface onto the latter will not preserve all aspects of the surface geometry. Perhaps the most natural mapping is of the rotating sphere surface onto the surface of a nonrotating sphere by $(\theta,\phi) \rightarrow (\theta' = \theta, \phi' = \phi)$, where (θ', ϕ') are spherical polar coordinates for the nonrotating sphere. In this representation, the trajectories of particle and geodesic arcs, regarded as sets of points (θ,ϕ), are faithfully preserved, but neither angles between trajectories nor distances between points are faithfully recorded. Alternatively, one could construct a representation on a nonrotating sphere which preserves a single point on the trajectory, (θ_0, ϕ_0) say, and the local angle ζ between the arc and the concurrent line of longitude as defined by $v_\theta = v \cos\zeta, v_\phi = v \sin\zeta$; but such a map will not, of course, preserve the coordinates (θ,ϕ) of individual points on the arc. These two representations are analogous to the two different representations of free particle and photon trajectories on the surface of a rotating disk discussed in MM. Actually, when $\alpha \leqslant 3^{-1/2}$ we can in principle do better than either of these: we show in Section 7 how to find a surface of revolution embedded in Euclidean 3-space which has the *same* intrinsic geometry as that of a rotating sphere, and which therefore provides a possible representation of trajectories preserving *all* geometrical features such as points, angles, and distances.

Second, it will usually be necessary to map in turn the intermediate nonrotating sphere or surface of revolution onto a Euclidean plane. This is a more conventional problem and will not be discussed here any further. In Figure 1, we have chosen the trajectory-preserving map onto the nonrotating sphere, followed by a stereographic projection of the sphere onto a Euclidean plane from the projection point $r' = 10a$, $\theta' = \pi/2$, $\phi' = 0$. This arrangement, modified if required to allow for a different projection point, is probably the most convenient for the representation of particle and photon trajectories since, as equation (3.14) shows, each trajectory on such a map is determined by the value of the single parameter $\omega a/\bar{v}$ (in addition to θ_0, ϕ_0, and σ). By contrast the theoretically superior mapping involving an intermediate surface of revolution requires knowledge of the individual values of $\alpha=\omega a/c$ *and* of $\bar{v}/c$, quite apart from the restriction to $\alpha \leqslant 3^{-1/2}$.

For convenience, we include in Figure 1 the projections of some lines of latitude and longitude, and we show in dotted form those sections of paths which lie on the "far" side of the sphere. Two points of interest are illustrated in these pictures: (i) the tendency for $\sigma=-1$ orbits to lie closer to the equator than corresponding $\sigma=+1$ orbits, for a fixed choice of end points, and (ii) that $d\phi/dt$ can sometimes (Figure 1c) alternate in sign along a trajectory, unlike $d\bar{\phi}/d\bar{t}$.

4. TIME OF FLIGHT BETWEEN FIXED REFERENCE POINTS

We begin by asking how long a free particle moving with speed $\bar{v}$ would take to travel between two points $(\bar{\theta}_1,\bar{\phi}_1)$ and $(\bar{\theta}_2,\bar{\phi}_2)$ on the nonrotating spherical surface $\bar{r}=a$. The answer can be provided analytically, starting from equations (3.3) and (3.5), but the quickest procedure is to apply equation (3.7) to the spherical triangle whose vertices are $(\bar{\theta}_1,\bar{\phi}_1)$, $(\bar{\theta}_2,\bar{\phi}_2)$ and the N pole (Figure 2). The great circle distance between the two end points on the trajectory must be $\bar{v}(\bar{t}_2-\bar{t}_1)$, where $\bar{t}_1$ and $\bar{t}_2$ are the times at which the particle coincides with the end points. Letting $A=\bar{\phi}_2-\bar{\phi}_1$, $\tilde{a}=\bar{v}(\bar{t}_2-\bar{t}_1)/a$, $\tilde{b}=\bar{\theta}_1$, $\tilde{c}=\bar{\theta}_2$, equation (3.7) yields

$$\cos\bar{v}\left(\bar{t}_2-\bar{t}_1\right)/a=\cos\bar{\theta}_1\cos\bar{\theta}_2+\sin\bar{\theta}_1\sin\bar{\theta}_2\cos\left(\bar{\phi}_2-\bar{\phi}_1\right) \tag{4.1}$$

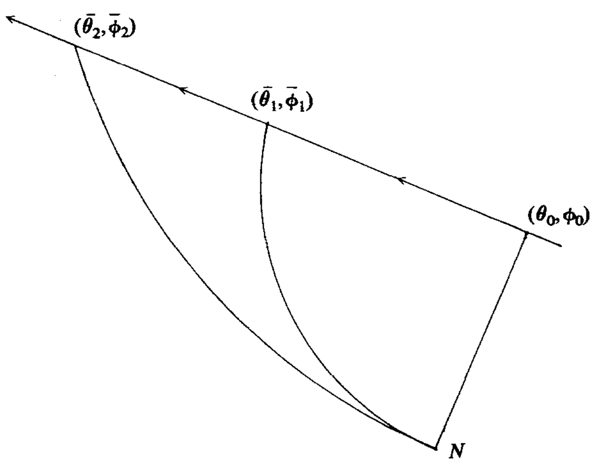

Fig. 2. Spherical triangles based on spatial path traced by a particle on the nonrotating sphere. N is the North pole and (θ_0,ϕ_0) is the point on the trajectory locally nearest the pole.

We deduce from the coordinate transformation (2.2) that the coordinate time taken (t_2-t_1) for the same free particle to travel from (θ_1,ϕ_1) to (θ_2,ϕ_2) on the surface of the rotating sphere satisfies

$$\cos\bar{v}(t_2-t_1)/a=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\left[\phi_2-\phi_1+\omega(t_2-t_1)\right] \quad (4.2)$$

For some special cases, an analytic solution to this equation exists. Thus, when the first point is the N pole $(\theta_1=0)$, we have

$$\bar{v}(t_2-t_1)/a=\theta_2+2n\pi$$

or

$$\bar{v}(t_2-t_1)/a=-\theta_2+2(n+1)\pi, \qquad n=0,1,2\ldots \quad (4.3)$$

where n is the integer part of the number of orbits traversed around the sphere by the particle before encountering the second point. Similar results emerge for the special cases $\theta_2=0$, $\theta_1=\pi$, or $\theta_2=\pi$; likewise, an equatorial orbit $(\theta_1=\theta_2=\pi/2)$ gives rise to equally simple solutions with an obvious physical interpretation.

For the general case, an analytic solution is not possible, but we can nevertheless make the following observations. First, interchange of the coordinates (θ_1,ϕ_1) and (θ_2,ϕ_2) does not leave equation (4.2) unchanged. Thus, in general, the time taken to pass between two reference points A and B depends not only on $\bar{v}$ and on where the reference points are located but also on the direction of travel ($A\to B$ or $B\to A$). Second, if $\omega a/\bar{v}$ is a rational number, m/n, say, where m and n are positive integers, then $t_2-t_1+n2\pi a/\bar{v}$ is a solution if t_2-t_1 is a solution. This corresponds to the case where the motion repeats itself after n orbits (in the θ sense), as discussed in Section 3. Third, if $\bar{v}$ is a solution to equation (4.2) for a fixed value of t_2-t_1, so also is $\bar{v}\pm n2\pi a/(t_2-t_1)$, where $n=1,2,3\ldots$, provided the latter does not exceed c or become negative. Physically this corresponds to the fact that there exists a lowest speed which a particle must have in order to pass from (θ_1,ϕ_1) to (θ_2,ϕ_2) in time t_2-t_1; but there are also faster particles which achieve the same end by adding on an integer number of extra orbits during the fixed time available.

From equation (4.2) we see that, for any choice of ϕ_1 and ϕ_2, the right-hand side has a minimum possible value of $\cos(\theta_1+\theta_2)$ and a maximum possible value of $\cos(\theta_1-\theta_2)$. This implies that the allowed values of $\bar{v}(t_2-t_1)/a$ must lie within a number of bands along the positive real axis. The first band runs from $|\theta_2-\theta_1|$ to the smaller of $\theta_1+\theta_2$, $2\pi-\theta_1-\theta_2$; the extremities both correspond to polar orbits, the first of which involves no intersection with a pole during the time t_2-t_1, the second involving one intersection with the N pole (if $\theta_1+\theta_2<\pi$) or the S

pole (if $\theta_1+\theta_2>\pi$). The second band runs from the greater of $\theta_1+\theta_2$, $2\pi-\theta_1-\theta_2$ to $2\pi-|\theta_2-\theta_1|$, the first extremity corresponding to a polar orbit intersecting the N pole (if $\theta_1+\theta_2>\pi$) or the S pole (if $\theta_1+\theta_2<\pi$) once, and the second to a polar orbit intersecting both poles once. The interpretation is similar for higher bands, which clearly correspond to longer and more complicated trajectories during the coordinate time interval t_2-t_1.

Figure 3 shows the relationship between $\bar{v}$ and t_2-t_1, for the particular choice $\theta_1=\pi/4$, $\phi_1=-\pi/6$ and $\theta_2=\pi/3$, $\phi_2=\pi/6$. The nondimensional variables used are $\beta=\bar{v}/\omega a$ and $\tau=\omega(t_2-t_1)$, and the figure exhibits some of the features discussed above. In particular, the branch of the multivalued function $\beta(\tau)$ nearest the origin corresponds to the first band of allowed values for the product $\beta\tau$, the second nearest to the second band, and so on. Variation of ϕ_1 and ϕ_2, for fixed θ_1 and θ_2, would cause systematic changes in the curves displayed, but always subject to the prohibition on $\beta\tau$ taking on any value in the forbidden regions between the allowed bands.

It is interesting to observe that $\beta(\tau)$ possesses stationary values for comparatively small values of β. This implies that, even within a single band, there is often more than one way in which a free particle with speed $\bar{v}$ can go from (θ_1,ϕ_1) to (θ_2,ϕ_2), the various possibilities involving different initial directions of motion and transit times. It can be shown, however, by a calculation analogous to that carried out in MM for a rotating disk, that β has no stationary values for which $\beta>1$. This implies that for a light ray, for which $\beta=c/\omega a=\alpha^{-1}$, there is only one solution for t_2-t_1 in each band, for a given value of β. Clearly the light ray solution for t_2-t_1 in the first band is the minimum possible time for a signal to pass from the first point to the second point.

To conclude this section, we show how the orbit parameters θ_0, ϕ_0, and σ may be determined, starting from knowledge of the end points (θ_1,ϕ_1), (θ_2,ϕ_2) and the particle velocity $\bar{v}$. Applying equation (3.9) to each of the two smaller spherical triangles in Figure 2, and combining the results with the coordinate transformation (2.2), we find

$$\sin\theta_0=\frac{\sin\theta_1\sin\theta_2\sin[\phi_2-\phi_1+\omega(t_2-t_1)]}{\sin\sigma\bar{v}(t_2-t_1)/a} \tag{4.4}$$

We can determine numerically the various allowed values of t_2-t_1 from equation (4.2), then substitute each in turn into (4.4) to find the corresponding values of θ_0 and (since the right-hand side must be nonnegative) of σ. The choice of any *one* of the variables t_1, t_2, and ϕ_0 is to some extent arbitrary, since ϕ_0 is not uniquely determined by the condition that θ

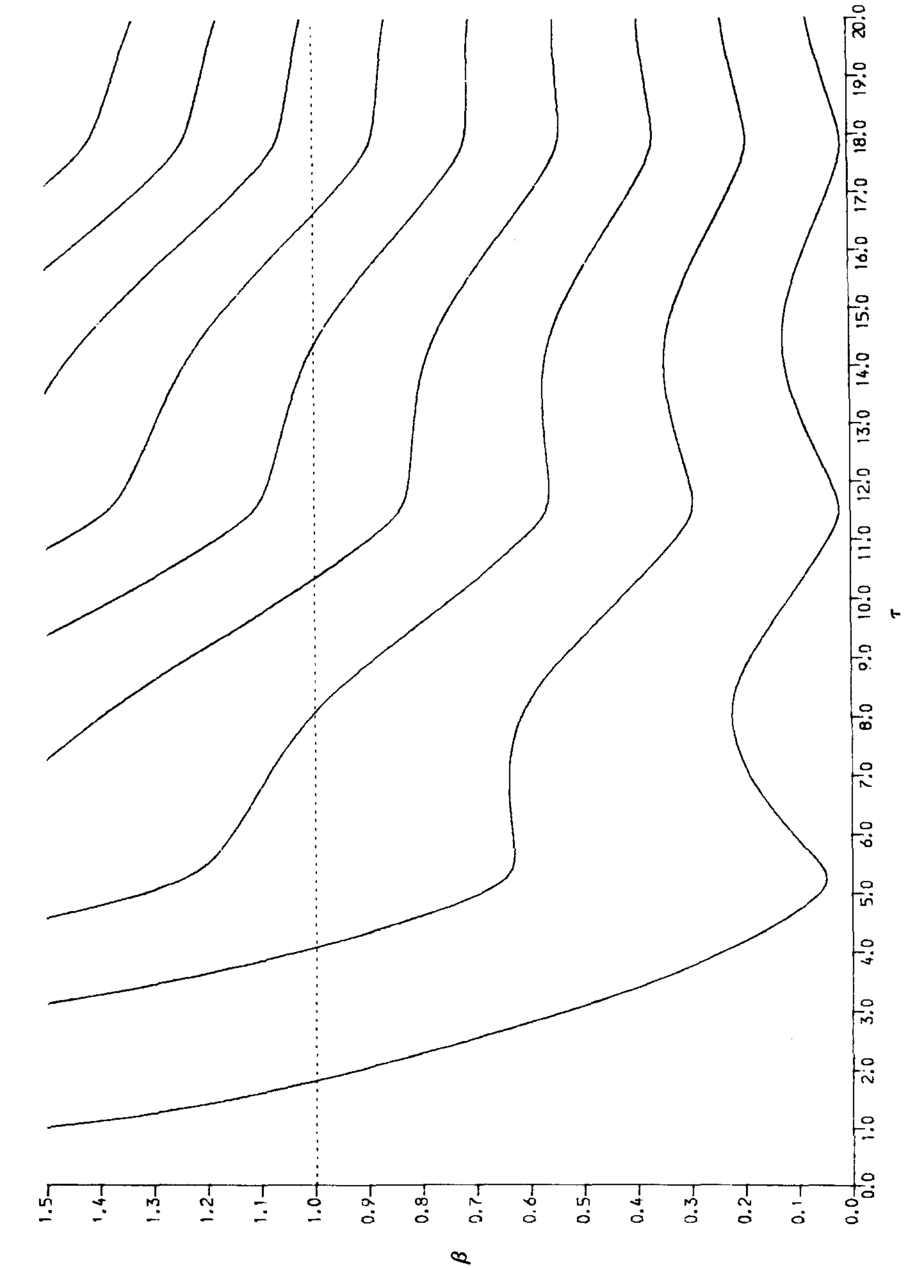

Fig. 3. Relation between velocity variable β and time-of-flight variable τ for the route $(\pi/4, -\pi/6) \rightarrow (\pi/3, \pi/6)$.

assumes its minimum value at (θ_0, ϕ_0)—unlike the case of a nonrotating sphere. Probably the most natural procedure is to set $\theta = \theta_1$, $t = t_1$, and $\theta = \theta_2$, $t = t_2$ in turn in equation (3.11) and to select—somewhat arbitrarily—a pair of solutions for t_1 and t_2 for which $t_2 - t_1$ is equal to the chosen solution of (4.2). Then ϕ_0 can be calculated by solving equations (3.12a) and (3.12b), having first inserted (say) $\theta = \theta_1$, $\phi = \phi_1$, $t = t_1$. This procedure was used in computing the orbits shown in Figure 1, all of which represent solutions lying in the first band for which the transit time between the chosen end points is least.

5. SPATIAL DISTANCE ALONG NULL GEODESICS

The spatial distance covered on the rotating sphere by a particle following a space-time geodesic—starting say from the point (θ_0, ϕ_0)—may be calculated by integrating the spatial line element in equation (2.4) (with $r = a, dr = 0$) along the locus described by equation (3.14). In practice, the required integral is fairly complicated and recourse must be made to numerical integration. For a photon, however, substitution of $\bar{v} = c$ leads to a much simpler integral following cancellation of terms, and we find

$$l(\theta) = a(1 - \sigma\alpha\sin\theta_0)\int_{\theta_0}^{\theta}(1 - \alpha^2\sin^2\theta)^{-1/2}\left(1 - \sin^2\theta_0/\sin^2\theta\right)^{-1/2} d\theta$$

$$= a(1 - \sigma\alpha\sin\theta_0)^{1/2}(1 + \sigma\alpha\sin\theta_0)^{-1/2}F(\chi|\lambda) \tag{5.1}$$

where

$$\chi = \cos^{-1}(\cos\theta/\cos\theta_0), \qquad \sin\lambda = \alpha\cos\theta_0\left(1 - \alpha^2\sin^2\theta_0\right)^{-1/2} \tag{5.2}$$

and F is the elliptic integral of the first kind,

$$F(\chi|\lambda) = \int_0^{\chi}\left(1 - \sin^2\lambda\sin^2\phi\right)^{-1/2} d\phi \tag{5.3}$$

This formula yields the distance from (θ_0, ϕ_0) to any point on the null geodesic up to the first crossing of the equator. Clearly the distance to points further away than this is obtained by adding on multiples of $l(\pi/2)$ to $\pm l(\theta)$ or $\pm l(\pi - \theta)$. Noting from equation (3.11) that $\chi = ct/a$, equation (5.1) also provides the distance traveled on the rotating sphere by a photon in time t.

It is interesting to note some special cases. For a polar orbit, $\theta_0 = 0$ and our result simplifies to $l(\theta) = aF(\theta|\sin^{-1}\alpha)$. In particular, the distance between the N or S pole and the equator along a light ray is $aF(\pi/2|$

$\sin^{-1}\alpha$), which is of course greater than the minimum possible distance between them, i.e., $\pi a/2$.[1] For an equatorial orbit ($\theta=\theta_0=\pi/2$), equation (5.1) is not applicable, but we can immediately integrate equation (2.4) to give $l(\phi)=a(\phi-\phi_0)(1-\alpha^2)^{-1/2}$.

The distance between two fixed points (θ_1,ϕ_1) and (θ_2,ϕ_2) along a light ray joining them is either $l(\theta_1)+l(\theta_2)$ or $|l(\theta_1)-l(\theta_2)|$, depending on whether (θ_0,ϕ_0) lies between them on the trajectory or not. In Section 4 we saw that each of the allowed values of t_2-t_1, and hence of θ_0, depends in general on the direction of the signal between the fixed end points as well as on their direction, and so it is sufficiently clear that this distance also depends, in general, on whether the light signal goes from (θ_1,ϕ_1) to (θ_2,ϕ_2) or in the opposite direction. This conclusion is the same as in the corresponding analysis for the rotating disk (MM).

6. SPATIAL GEODESICS

Setting $r=a$ in equation (2.4), the line element dl on the surface of a rotating sphere satisfies

$$dl^2=a^2\left(d\theta^2+\frac{\sin^2\theta\, d\phi^2}{1-\alpha^2\sin^2\theta}\right) \tag{6.1}$$

Spatial geodesics, i.e., paths along which the distance between any two fixed points (within certain limits—see below) is a minimum, are obtained by solving the geodesic equation (3.1) with ds replaced by dl and metric given by equation (6.1). With $\kappa=1,2$ in turn, we get

$$\frac{d^2\theta}{dl^2}-\frac{\sin\theta\cos\theta}{(1-\alpha^2\sin^2\theta)^2}\left(\frac{d\phi}{dl}\right)^2=0, \qquad \frac{d\phi}{dl}=\frac{k(1-\alpha^2\sin^2\theta)}{\sin^2\theta} \tag{6.2}$$

where k is constant. If $\theta(l)$ possesses a stationary (minimum) value θ_0 say ($\neq 0$), equation (6.1) yields an expression for $d\phi/dl$ at $\theta=\theta_0$, which can be equated to the corresponding expression in (6.2) evaluated at the same point. For $k\neq 0$ we deduce

$$k=(\gamma_0/a)\sin\theta_0 \tag{6.3}$$

where

$$\gamma_0=\left(1-\alpha^2\sin^2\theta_0\right)^{-1/2} \tag{6.4}$$

[1] The latter result follows from equation (6.8).

Eliminating $d\phi/dl$ from the two equations in (6.2), integration of the equation for θ gives

$$\cos\theta = \cos\theta_0 \cos(\gamma_0 l/a) \tag{6.5}$$

where, for convenience, arc length is measured from $\theta = \theta_0$. Integration of the equation for ϕ, with initial condition $\phi = \phi_0$ when $\theta = \theta_0$, yields the corresponding solution

$$\sin\theta\cos\left[\phi - \phi_0 + \alpha^2(\gamma_0 l/a)\sin\theta_0\right] = \sin\theta_0 \cos(\gamma_0 l/a) \tag{6.6a}$$

$$\sin\theta\sin\left[\phi - \phi_0 + \alpha^2(\gamma_0 l/a)\sin\theta_0\right] = \sin(\gamma_0 l/a) \tag{6.6b}$$

In these equations it is assumed that l is a *signed* length, increasing in the direction of increasing ϕ. It should be noted that the equator, described by

$$\theta = \theta_0 = \pi/2, \qquad \phi - \phi_0 = (1-\alpha^2)^{1/2} l/a \tag{6.7}$$

is a special case of the general solution; polar geodesics on the other hand are singular solutions for which $k=0$, and are polar great circles represented by

$$\cos\theta = \cos l/a, \qquad \phi - \phi_0 = 0 \text{ or } \pi \tag{6.8}$$

Clearly these solutions are the spatial analogs of the solutions for space-time geodesics in equations (3.11)–(3.13), the variable $\gamma_0 l/a$ playing the same role as $\sigma\bar{v}t/a$. Eliminating $\gamma_0 l/a$ from equations (6.5) and (6.6), the general spatial geodesic (excluding the equatorial solution) may be expressed in the alternative form

$$\phi - \phi_0 = \pm\left[\cos^{-1}(\tan\theta_0/\tan\theta) - \alpha^2\sin\theta_0\cos^{-1}(\cos\theta/\cos\theta_0)\right] \tag{6.9}$$

which is the analog of equation (3.14). A free particle or photon can thus follow a spatial geodesic route only if $\sigma = 1$ and if $\alpha^2 \sin\theta_0 = \omega a/\bar{v}$; this would require $\bar{v} = c(\alpha \sin\theta_0)^{-1}$, a condition plainly impossible to fulfill. Hence the locus of points traced out by a free particle or photon on the surface of a rotating sphere does not coincide with a spatial geodesic, except of course for the special case of an equatorial orbit.

It is apparent from equation (6.9) that, starting from (θ_0, ϕ_0) and proceeding along the spatial geodesic in either direction, the equator is first intersected at points satisfying

$$\phi - \phi_0 = \pm\tfrac{1}{2}\pi(1 - \alpha^2\sin\theta_0) \tag{6.10}$$

and that, necessarily interpreting $\cos^{-1}$ as a multivalued function which continuously increases in value as we move around the sphere, further intersections of the equator occur when $\phi-\phi_0$ is an odd integer times the right-hand side of (6.10). The spatial geodesic will be closed if, after n complete orbits in the θ sense, the change in ϕ is $m2\pi$ (where m and n are integers), i.e., if $1-\alpha^2 \sin\theta_0 = m/n$. Otherwise the spatial geodesic never closes.

The previous paragraph reveals the existence of multiple geodesic paths (in the sense, at least, of solutions to the geodesic equation) connecting pairs of points on the sphere. Consider for example two points on the equator. Equation (6.10) shows that the angular displacement in the ϕ direction between two successive crossings of the equator by a spatial geodesic is never less than $\pi(1-\alpha^2)$. Suppose then that the angular separation $\Delta\phi$ for our two fixed points satisfies

$$\Delta\phi \leqslant \pi, \qquad r < \frac{\Delta\phi}{\pi(1-\alpha^2)} \leqslant r+1 \tag{6.11}$$

where r is a positive integer or zero. Discounting paths involving multiple revolutions round the sphere and paths which join the two points "the long way round," there will be $r+1$ distinct paths between them which are solutions of the geodesic equation. One of these is the equatorial route, along which the intervening distance is

$$l_0 = a\Delta\phi(1-\alpha^2)^{-1/2} \tag{6.12}$$

The others may be parametrized by an integer j ($j=1,2,\ldots,r$), where $j-1$ is the number of times the path intersects the equator *between* the end points. For each value of j there will be a separate value of the parameter θ_0, θ_{0j} say, obtained from

$$\Delta\phi = j\pi\left(1-\alpha^2 \sin\theta_{0j}\right), \qquad j=1,2,\ldots,r \tag{6.13}$$

Equation (6.5) implies that the distance along the geodesic path between each successive pair of equatorial intersections is $\pi a(1-\alpha^2\sin^2\theta_{0j})^{1/2}$, and so the total distance between our two points along this route is

$$l_j = j\pi a\left(1-\alpha^2\sin^2\theta_{0j}\right)^{1/2}, \qquad j=1,2,\ldots,r \tag{6.14}$$

An algebraic calculation now shows that $l_0 > l_1$ and that $l_j > l_{j-1}$ for $j=2,3,\ldots,r$. (Alternatively the latter result can be achieved by examining the derivative of l_j with respect to j, treating j as a continuous variable.)

Thus the shortest distance between two points on the equator is obtained by following the equatorial route between them if $\Delta\phi \leqslant \pi(1-\alpha^2)$; and by following the "one-hop" geodesic route between them if $\Delta\phi$ lies outside this range.

More generally, it would be useful to extend this analysis to two *arbitrary* points on the sphere, and to be able to identify the geodesic which provides the shortest distance between them. We are not able to give a complete solution to this problem, but we have devised the following procedure which will allow the geodesic paths of greatest interest to be ascertained and examined, for any choice of end points. We begin by calculating the various allowed values of the parameter θ_0, for fixed end points, similar to the procedure outlined in Section 4 for space-time geodesics. Here we can make use of the analogy between the solutions of the spatial and space-time geodesic equations and simply write down equations that correspond to the time-of-flight equation (4.2) and equation (4.4) for $\sin\theta_0$; alternatively, these equations can be proved by eliminating ϕ_0 from equations (6.5) and (6.6), having first inserted the coordinates of the end points. By either route we find

$$\cos\varepsilon = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1+\alpha^2\varepsilon\sin\theta_0) \tag{6.15}$$

$$\sin\theta_0 = \sin\theta_1\sin\theta_2\sin(\phi_2-\phi_1+\alpha^2\varepsilon\sin\theta_0)/\sin\varepsilon \tag{6.16}$$

where

$$\varepsilon = \gamma_0(l_2-l_1)/a \tag{6.17}$$

and l_2-l_1 is the distance between the end points along the route characterized by θ_0. At this stage, however, the analogy breaks down: for fixed α, equation (6.15) cannot be used to find the allowed values of ε from knowledge of the coordinates (θ_1, ϕ_1) and (θ_2, ϕ_2) alone, due to the appearance of the extra $\sin\theta_0$ factor on the right-hand side. The best way to proceed is to combine equations (6.15) and (6.16), yielding

$$\sin^2\theta_0\sin^2\varepsilon = \sin^2\theta_1\sin^2\theta_2 - (\cos\varepsilon - \cos\theta_1\cos\theta_2)^2 \tag{6.18}$$

This provides an expression for $\sin\theta_0$ in terms of ε, which on substitution into equation (6.15) yields a single equation for the unknown quantity ε. Having solved this equation numerically, we then use equations (6.18) and (6.17) to calculate the corresponding solutions for $\sin\theta_0$ and l_2-l_1.

Whatever the values of ϕ_2, ϕ_1, or ε, we see from equation (6.15) that

$$\cos(\theta_1+\theta_2) \leqslant \cos\varepsilon \leqslant \cos(\theta_1-\theta_2) \tag{6.19}$$

so that the allowed values of ε are confined to the same bands of possible values as for $\bar{v}(t_2-t_1)/a$ in the corresponding problem for space-time geodesics. Defining

$$f(\varepsilon)=\cos\varepsilon-\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1+\alpha^2\varepsilon\sin\theta_0) \quad (6.20)$$

it is readily confirmed that $f(\varepsilon)\geqslant 0$ at the upper limit for cos ε and $f(\varepsilon)\leqslant 0$ at the lower limit. Since $f(\varepsilon)$ is a continuous function, there is at least one root of the equation $f(\varepsilon)=0$ within each band, and this can be found by direct numerical evaluation of $f(\varepsilon)$. It should be noted that, because of the presence of the factor γ_0 in equation (6.17), an ascending sequence of allowed values of ε does not necessarily generate the allowed values of l_2-l_1 in correct ascending order. Nevertheless, since γ_0 is never greater than $(1-\alpha^2)^{-1/2}$, the allowed values of ε in the higher bands *must* eventually correspond to paths involving multiple revolutions around the sphere, so that the search for the geodesic solution which gives the minimum distance is never likely to be protracted. Indeed we conjecture that the shortest path between two points always corresponds to a solution for ε in the first band, partly because this is certainly true in the two extreme cases where $\phi_1=\phi_2$ (i.e., a polar geodesic with $l_2-l_1=a|\theta_2-\theta_1|$ as the first band solution) and where $\phi_1=\phi_2+\pi$ [a polar geodesic with $l_2-l_1=a\times\mathrm{Min}(\theta_1+\theta_2, 2\pi-\theta_1-\theta_2)$ as the first band solution]. We also conjecture that there is only one solution to $f(\varepsilon)=0$ per band, although this is supported only by practical experience in the few cases which we have explicitly investigated.

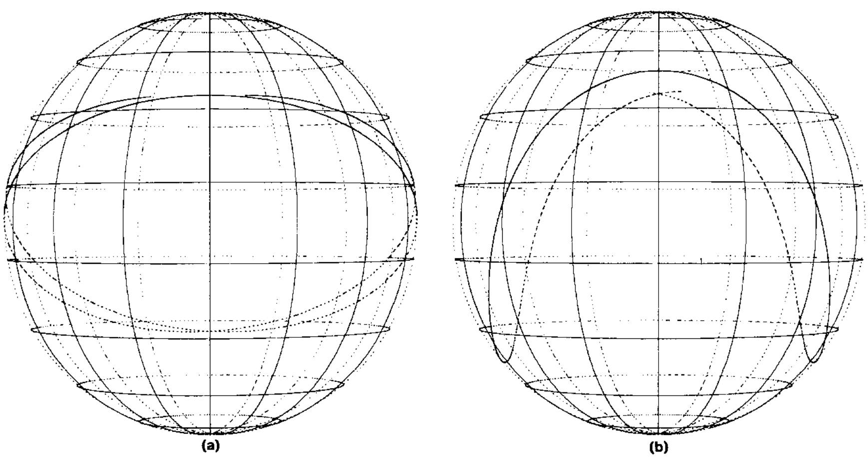

Fig. 4. Spatial geodesics joining A and B of Figure 1 with (a) $\alpha=0.2$ and (b) $\alpha=0.8$.

Having determined θ_0 and $l_2 - l_1$ by the above procedure, the calculation of the individual values of ϕ_0, l_1, and l_2 is exactly analogous to that described in Section 4 for the determination of ϕ_0, t_1, and t_2, and will not be further discussed. Figure 4 displays spatial geodesics, represented as for Figure 1, passing through the points $(\pi/3, -\pi/6)$ and $(\pi/3, \pi/6)$ for $\alpha = 0.2$ and $\alpha = 0.8$. In each case the geodesic shown is that which gives the shortest distance between the end points. For $\alpha = 0.2$, the geodesic path does not differ very much (locally, at least) from a great circle, but a substantial difference may be noted for the higher value of α.

7. EMBEDDING THE SPHERICAL GEOMETRY IN A FLAT 3-SPACE

If we define

$$\rho = a \sin\theta (1 - \alpha^2 \sin^2\theta)^{-1/2} \tag{7.1}$$

the line element given by equation (6.1) assumes the simple appearance

$$dl^2 = a^2 d\theta^2 + \rho^2 d\phi^2 \tag{7.2}$$

We can regard this as one form of the line element of a surface of revolution $z = z(\rho)$ embedded in a Euclidean 3-space in which (ρ, ϕ, z) are cylindrical polar coordinates. Thus if we alternatively write

$$dl^2 = dz^2 + d\rho^2 + \rho^2 d\phi^2 \tag{7.3}$$

then combine the expressions for dl^2 and differentiate equation (7.1) with respect to θ, we find the following expression for z in terms of the parameter θ:

$$z = a \int_\theta^{\pi/2} \left[1 - \frac{\cos^2\theta}{(1 - \alpha^2 \sin^2\theta)^3} \right]^{1/2} d\theta \tag{7.4}$$

The selected boundary condition is that $z = 0$ corresponds to the equatorial plane of the sphere ($\theta = \pi/2$). Elimination of θ between equations (7.1) and (7.4) yields the surface $z = z(\rho)$. By construction, this surface has the same intrinsic geometry as the surface of the rotating sphere.

In order for the embedded surface to exist for all relevant values of θ ($0 \leqslant \theta \leqslant \pi$), it is necessary that the right-hand side of equation (7.4) be real throughout this range. Inspection of the integrand shows that it is always real if $\alpha^2 \leqslant 1/3$, but becomes complex in the vicinity of $\theta = 0$ and $\theta = \pi$ for

$\alpha^2 > 1/3$. This means that parts of the embedded surface near the poles cease to exist as α^2 increases beyond $1/3$, and as $\alpha \to 1$ the angular fraction of the rotating spherical surface represented shrinks to zero.

It is possible to express the right-hand side of equation (7.4) as a rather complicated combination of elliptic integrals of the first, second, and third kinds, but it is simpler to carry out the integration numerically for any particular value of α. Figure 5 shows the results for $\alpha^2 = 0$, $1/12$, and $1/3$, where in each case z/a is plotted against ρ/a, the dotted lines indicating the values of the parameter θ at 9° intervals. As expected, the embedded surface for $\alpha^2 = 0$ is itself spherical, but as α^2 increases it becomes progressively more oblate. For $\alpha^2 = 1/3$, the curvature of the surface at $\theta = 0$ and $\theta = \pi$ is zero, as the figure suggests and as a calculation based on equation (6.1) confirms. It is worth emphasizing that the variable θ, though it retains its meaning as a spherical polar coordinate in the frame of the rotating sphere, is *not* a spherical polar coordinate in the Euclidean 3-space in which the surface $z(\rho)$ is embedded. In the latter space its main significance, as equation (7.2) and Figure 5 demonstrate, is that it varies

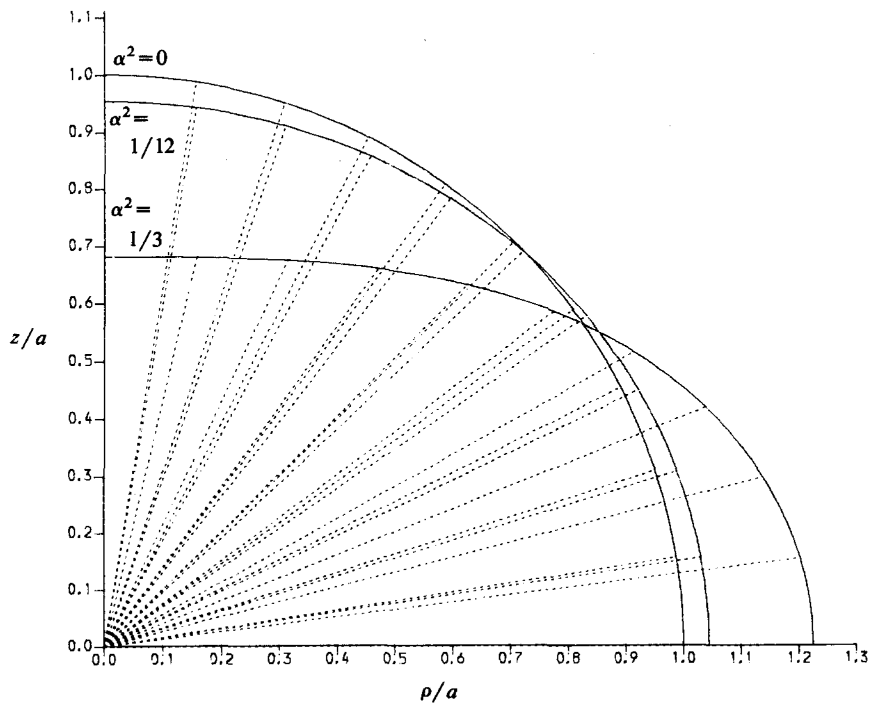

Fig. 5. Cross section $z = z(\rho)$ of embedding surface for $z > 0$. The embedding surface for negative z is the mirror image of this.

linearly with distance traveled on the embedded surface along any path for which $d\phi=0$. Points in the Euclidean 3-space that do not lie on the embedded surface have no physical significance.

The embedding is useful in providing a visual appreciation of some of the features of the spatial geometry of a rotating sphere discussed previously. Consider again, for example, the problem of finding the shortest path between two points on the equator, and suppose we employ a stretched string to identify paths that are solutions of the spatial geodesic equation. It is intuitively obvious, by reference to the embedding displayed in Figure 5, that a stretched string joining two points on the equator by the route of minimum distance will itself lie on the equator provided the angular separation (in the ϕ direction) between them is small. As the angular separation increases, however, the equatorial configuration will eventually become unstable and the string, if dispaced slightly, will slip over the surface until it reaches the configuration corresponding to the alternative (stable) geodesic path wholly within one of the two hemispherical surfaces.

ACKNOWLEDGMENTS

Two of us (KM and IHM) acknowledge the financial support of SRC Research Studentships.

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